Uzawa's (1961) steady-state growth theorem

In a neoclassical growth model, the existence of a balanced growth requires that either

- the production function is Cobb-Douglas, or
- technological progress is purely labor-augmenting.

(Note: "neoclassical growth model"

$$Y = F(K, L, A) \qquad \dot{K} = Y - C - \delta K$$

has CRS with respect to K, L $\qquad L(t) = L_0 e^{nt}$)

(Note #2: "balanced growth path" means that:

$$Y(t) = Y_0 e^{yt}, \quad C(t) = C_0 e^{ct}, \quad K(t) = K_0 e^{kt}$$)

Proof (Schlicht, 2006)

- Write $Y(t) = F(K(t), L(t), t)$.
- From the equation of motion, we have

$$\dot{K}(t) = k\, K(t) = Y(t) - C(t) - \delta K(t)$$

$$\Rightarrow (k+\delta) K_0 = Y_0 e^{(y-k)t} - C_0 e^{(c-k)t}, \quad \text{for all } t \geq 0.$$

- Taking time derivatives,

$$(y-k) Y_0 e^{(y-k)t} - (c-k) C_0 e^{(c-k)t} = 0$$

$$(y-k) Y_0 e^{(y-c)t} = (c-k) C_0$$

- Therefore either $\underbrace{y = k \text{ and } c = k}$, or $\underbrace{y = c \text{ and } Y_0 = C_0 \text{ (and hence } K_0 = 0)}_{\text{CONTRADICTION.}}$

[GROWTH RATES OF Y, C, K COINCIDE!

- Define $G(K,L) := F(K,L,0)$.

We have $Y_0 = G(K_0, L_0)$, $Y(t) = Y_0 e^{yt}$, $L_0 = L(t)e^{-nt}$, $K_0 = K(t)e^{-kt}$, and G is linear homogenous.

- Hence $Y(t) = G(K_0, L_0)e^{yt} = G\left(K(t)e^{-kt}, L(t)e^{-nt}\right)e^{yt} =$

$$= G\left(K(t)e^{(y-k)t}, L(t)e^{(y-n)t}\right) = G\left(K(t), L(t)e^{(y-n)t}\right).$$

PURELY LABOR-AUGMENTING TC ∎

Comment. Where is the Cobb-Douglas ??

- Observe: if $Y(t) = \underbrace{A(t)}_{\text{any time trend}} \cdot K^{\alpha}(t) L^{1-\alpha}(t)$ then we can always rewrite it as $Y(t) = K^{\alpha}(t) \underbrace{(\tilde{A}(t)L(t))^{1-\alpha}}_{\text{LATC}}$ (here: $\tilde{A}(t) = A(t)^{\frac{1}{1-\alpha}}$)

$\downarrow$

OBSERVATIONALLY EQUIVALENT $\boxed{!}$

- However, we could also, e.g., write it as

$$Y(t) = \underbrace{(\tilde{\tilde{A}}(t)K(t))^{\alpha}}_{\text{KATC}} L(t)^{1-\alpha} \quad \left(\text{here: } \tilde{\tilde{A}}(t) = A(t)^{\frac{1}{\alpha}}\right).$$

- For other, non-multiplicative production fcts, alternative patterns of factor augmentation are not observationally equivalent.

Note #1: All growth models with endogenous technical change discussed so far featured Cobb-Douglas production functions. This was not only for simplification.

Note #2: Empirically, LATC is rather reasonable, given that
- rates of return to a unit of capital have been broadly stable over time
- wages have been rising roughly exponentially
- labor's share of GDP has been broadly stable over the long run

} Kaldor (1961) facts (NOW CONTESTED!)